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Lacunary interpolation by quartic splines on uniform meshes

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Abstract

The interpolation of a discrete set of data on the interval [0, 1], representing the first and the second derivatives (except at 0) of a smooth function f is investigated via quartic C^2-splines. Error bounds in the uniform norm for $\|s^{(i)} - f^{(i)}\|$, $i = 0(1)2$, if $f \in C^l[0,1]$, $l = 3,5$ and $f^{(3)} \in BV[0,1]$, together with computational examples will also be presented.

Keywords: Lacunary interpolation; Quartic splines

AMS classification: 41A15; 65D07

1. Introduction

This paper presents the construction and the convergence properties of quartic C^2-splines interpolating smooth functions on uniform meshes with mesh size h. An attempt has been made to extend and improve the results presented in [7] by using quartic spline interpolant with periodic second derivative, but unfortunately no unique interpolant exists. However, when periodicity requirement is relaxed uniqueness is guaranteed.

In Section 2, the existence and uniqueness of such quartic splines is proved, if $f \in C^l[0,1]$, $l = 3,5$ and $f^{(3)} \in BV[0,1]$. Similar results can also be achieved for $l = 2,4$ by using similar arguments. Section 3 contains our main results where error bounds in L_∞-norm for $\|s^{(i)} - f^{(i)}\|$, $i = 0(1)2$ are derived. We conclude with numerical test examples in Section 4. Related procedures for periodic quartic and quintic splines using the nodal values $f_i (i = 0(1)N + 1)$ have been examined in [1–5, 9, 10, 12], and the references cited therein. Similar problems are also dealt with in [6, 8].

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2. Construction of a lacunary quartic spline

Let a uniform knot set $\{x_i\}_{i=0}^{N+1}$ be given with $x_0=0,\ x_{N+1}=1$, and $h=x_{i+1}-x_i,\ i=0(1)N$. Denote by $S_{N,4}^{(2)}$ the collection of all quartic splines $s(x)$ such that:

$$s\in C^2[0,1],\qquad s(x) \text{ is a quartic polynomial on each subinterval } [x_i,x_{i+1}].$$

In addition, we set $f_i^{(k)}=f^{(k)}(x_i),\ i=0(1)N+1,\ k\geqslant 0$.

Here we are concerned with the spline interpolation problem: Given the real numbers $\{f_i'\}_{i=0}^{N+1}$, $\{f_i''\}_{i=1}^{N+1}$ and f_0, f_{N+1}, there exists a unique $s\in S_{N,4}^{(2)}$ such that

$$s_0=f_0,\qquad s_{N+1}=f_{N+1},$$

$$s_i'=f_i'\quad (i=0(1)N+1),\qquad s_i''=f_i''\quad (i=1(1)N+1).$$

It can easily be verified that $s(x)$ restricted to $[x_i,x_{i+1}]$ will be written as:

$$s(x)=s_iA(t)+s_{i+1}B(t)+hf_i'C(t)+hf_{i+1}'D(t)+h^2s_i''E(t), \tag{2.1}$$

where

$$\begin{aligned} &A(t)=3t^4-4t^3+1,\qquad B(t)=4t^3-3t^4,\qquad C(t)=2t^4-3t^3+t,\\ &D(t)=t^4-t^3,\qquad E(t)=\tfrac{1}{2}t^4-t^3+\tfrac{1}{2}t^2, \end{aligned} \tag{2.2}$$

and $x=x_i+th,\ t\in[0,1]$, with a similar expression for $s(x)$ in $[x_{i-1},x_i]$.

Since $s\in C^2[0,1]$, then $s''(x_i^-)=s''(x_i^+)$ $(i=1(1)N)$ and this together with the condition $s_{N+1}''=f_{N+1}''$ lead to the following system in the unknown vector

$$\boldsymbol{S}=(s_1,s_2,\ldots,s_N,s_0'')^{\mathrm{T}}:$$

$$-s_{i-1}+s_i-\tfrac{1}{12}h^2(s_{i-1}''-s_i'')=\tfrac{1}{2}h(f_{i-1}'+f_i'),\qquad i=1(1)N+1. \tag{2.3}$$

This system has the coefficient matrix

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & -\frac{1}{12}h^2\\ -1 & 1 & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & -1 & 1 & 0\\ 0 & \cdots & \cdots & -1 & 0 \end{bmatrix}$$

whose determinant is obviously $-\frac{1}{12}h^2$ and hence s is uniquely determined. Using (2.3), one easily gets

$$s_0''=\frac{12}{h^2}(f_{N+1}-f_0)-\frac{6}{h}\left(f_0'+2\sum_{i=1}^{N}f_i'+f_{N+1}'\right)+f_{N+1}'', \tag{2.4}$$

and the other unknowns can be determined consecutively from the recurrence formula (2.3).

3. Approximation of smooth functions

In the sequel we write

$$BV[0,1]=\{f^{(3)}:[0,1]\to \mathbf{R}\,|\,Var(f^{(3)})<\infty\},$$

where $Var(f^{(3)})$ denotes the total variation of $f^{(3)}$ on [0,1] and $\omega_3(h)$ its modulus of continuity.

Let $e^{(i)}(x)=s^{(i)}(x)-f^{(i)}(x)$, $i=0(1)2$ and $e^{(i)}(x_j)=e_j^{(i)}$ $(j=0(1)N+1)$. The purpose of this section is to obtain a priori error estimates, in the uniform norm, for $e^{(i)}(x)$ if $f\in C^l[0,1]$, $l=3,5$ and $f^{(3)}\in BV[0,1]$ as stated in the following theorems.

Theorem 1. *Let $s(x)$ be the quartic spline defined in* (2.1) *and* (2.2), *then for $x\in[x_1,x_{N+1}]$, we have*

$$|e''(x)|\leqslant 6h\omega_3(h)\quad \textit{if } f\in C^3[0,1] \text{ or } f^{(3)}\in BV[0,1], \tag{3.1}$$

$$|e'(x)|\leqslant \tfrac{5}{2}h^2\omega_3(h)\quad \textit{if } f\in C^3[0,1] \text{ or } f^{(3)}\in BV[0,1], \tag{3.2}$$

and

$$|e(x)|\leqslant\begin{cases}\tfrac{1}{4}h^2(2+13h)\omega_3(h) & \textit{if } f\in C^3[0,1],\\ \tfrac{13}{4}h^3\omega_3(h)+\tfrac{1}{2}h^3\,Var(f^{(3)}) & \textit{if } f^{(3)}\in BV[0,1].\end{cases} \tag{3.3}$$

Moreover, if $x\in[x_0,x_1]$, then

$$|e''(x)|\leqslant\begin{cases}\tfrac{13}{2}(1+2h)\omega_3(h) & \textit{if } f\in C^3[0,1],\\ 13h\omega_3(h)+\tfrac{13}{2}h\,Var(f^{(3)}) & \textit{if } f^{(3)}\in BV[0,1],\end{cases} \tag{3.4}$$

$$|e'(x)|\leqslant\begin{cases}\tfrac{13}{2}h(1+2h)\omega_3(h) & \textit{if } f\in C^3[0,1],\\ 13h^2\omega_3(h)+\tfrac{13}{2}h^2\,Var(f^{(3)}) & \textit{if } f^{(3)}\in BV[0,1],\end{cases} \tag{3.5}$$

and

$$|e(x)|\leqslant\begin{cases}\tfrac{1}{4}h^2\,(2+3h)\,\omega_3(h) & \textit{if } f\in C^3[0,1],\\ \tfrac{3}{4}h^3\omega_3(h)+\tfrac{1}{2}h^3\,Var(f^{(3)}) & \textit{if } f^{(3)}\in BV[0,1].\end{cases} \tag{3.6}$$

Theorem 2. *Let $s(x)$ be the quartic spline defined in* (2.1) *and* (2.2). *If $f\in C^5[0,1]$, and $x\in[x_1,x_{N+1}]$, then*

$$|e''(x)|\leqslant \tfrac{1}{24}h^3\|f^{(5)}\|_\infty, \tag{3.7}$$

$$|e'(x)|\leqslant \tfrac{1}{384}h^4\|f^{(5)}\|_\infty, \tag{3.8}$$

and

$$|e(x)|\leqslant \tfrac{1}{180}h^4\,(42+5h)\,\|f^{(5)}\|_\infty. \tag{3.9}$$

Furthermore, if $x\in[x_0,x_1]$, then

$$|e''(x)|\leqslant \tfrac{13}{240}h^2\,(4+11h)\,\|f^{(5)}\|_\infty, \tag{3.10}$$

$$|e'(x)| \leqslant \tfrac{13}{240}h^3\,(4+11h)\,\|f^{(5)}\|_\infty, \tag{3.11}$$

$$|e(x)| \leqslant \tfrac{1}{720}h^4\,(12+31h)\,\|f^{(5)}\|_\infty. \tag{3.12}$$

To prove Theorems 1 and 2, we need the following lemma.

Lemma 3. *Let* $f \in C^3[0,1]$, *or* $f^{(3)} \in BV[0,1]$, *then*

$$|e_0''| \leqslant \begin{cases} \tfrac{1}{2}\omega_3(h), \\ \tfrac{1}{2}h\, Var\,(f^{(3)}). \end{cases} \tag{3.13}$$

Moreover, if $f \in C^5[0,1]$, *then*

$$|e_0''| \leqslant \tfrac{1}{60}h^2\|f^{(5)}\|_\infty. \tag{3.14}$$

Proof. Using (2.4), we have

$$\frac{h^2}{12}e_0'' = \int_0^1 f'(r)\,\mathrm{d}r - \frac{h}{2}\left(f_0' + 2\sum_{i=1}^{N} f_i' + f_{N+1}'\right) + \frac{h^2}{12}(f_{N+1}'' - f_0'').$$

Hence,

$$\frac{h^2}{12}e_0'' = \sum_{i=0}^{N}\left[\int_{x_i}^{x_i+1} f'(r)\,\mathrm{d}r - \frac{h}{2}(f_i' + f_{i+1}') - \frac{h^2}{12}(f_i'' - f_{i+1}'')\right].$$

If $f \in C^3[0,1]$, one gets

$$f'(x) = f_i' + (x - x_i)f_i'' + \tfrac{1}{2}(x - x_i)^2 f^{(3)}(\alpha_i),$$

and

$$f'(x) = f_{i+1}' + (x - x_{i+1})f_{i+1}'' + \tfrac{1}{2}(x - x_{i+1})^2 f^{(3)}(\beta_i), \qquad \alpha_i, \beta_i \in (x_i, x_{i+1}).$$

Integrating both sides of the first equation over $[x_i, x_i + \frac{1}{2}h]$ and those of the second over $[x_i + \frac{1}{2}h, x_{i+1}]$, using the mean value theorem for integrals and then adding, we obtain

$$\int_{x_i}^{x_{i+1}} f'(x)\,\mathrm{d}x = \frac{h}{2}\,(f_i' + f_{i+1}') + \frac{h^2}{8}\,(f_i'' - f_{i+1}'') + \frac{h^3}{48}[f^{(3)}(\alpha_i) + f^{(3)}(\beta_i)],$$

or

$$\int_{x_i}^{x_{i+1}} f'(x)\,\mathrm{d}x = \frac{h}{2}(f_i' + f_{i+1}') + \frac{h^2}{12}(f_i'' - f_{i+1}'') + \frac{h^3}{48}[f^{(3)}(\alpha_i) - 2f^{(3)}(\gamma_i) + f^{(3)}(\beta_i)], \tag{3.15}$$

where $\alpha_i, \beta_i, \gamma_i \in (x_i, x_{i+1})$. Thus,

$$e_0'' = \frac{h}{4}\sum_{i=0}^{N}[f^{(3)}(\alpha_i) - 2f^{(3)}(\gamma_i) + f^{(3)}(\beta_i)].$$

Consequently, if $f \in C^3[0,1]$, then

$$|e_0''| \leqslant \tfrac{1}{2}\omega_3(h),$$

and, if $f^{(3)} \in BV[0,1]$, then

$$|e_0''| \leqslant \tfrac{1}{2}h\, Var(f^{(3)}).$$

Furthermore, if $f \in C^5[0,1]$, then making use of the error of the corrected trapezoidal rule, we obtain

$$|e_0''| \leqslant \tfrac{1}{60}h^2 \, \|f^{(5)}\|_\infty .$$

We now turn to prove Theorems 1 and 2. For the sake of brevity, only estimates for error bounds if $f \in C^3[0,1]$ or $f^{(3)} \in BV[0,1]$ will be considered. Error bounds for $f \in C^5[0,1]$ can be handled in a similar manner. So the proof of Theorem 2 will be omitted.

Proof of Theorem 1. Let $f \in C^3[0,1]$, and $x \in [x_1, x_{N+1}]$ and using the fact that s' is a cubic Hermite interpolant for f' in $[x_i, x_{i+1}]$ $(i=1(1)N)$ (this is not the case in the subinterval $[x_0, x_1]$), it follows (see [11]), that

$$|e'(x)| \leqslant \tfrac{5}{2}h^2\omega_3(h) \quad \text{and} \quad |e''(x)| \leqslant 6h\omega_3(h).$$

For $x \in [x_0, x_1]$, we need e_1. This can be found by considering the Hermite–Birkhoff interpolating quartic polynomial $H(x)$ of $f(x)$ on $[x_0, x_1]$ based on the data $(x_0, f_0), (x_0, f_0'), (x_0, f_0''), (x_1, f_1'),$ (x_1, f_1''). Since $s(x)$ is the Hermite–Birkhoff interpolant for the data $(x_0, f_0), (x_0, f_0'), (x_0, s_0''), (x_1, f_1'),$ (x_1, f_1'') we have

$$|s(x) - H(x)| \leqslant |e_0''|\, h^2.$$

If we write

$$e(x) = s(x) - H(x) + H(x) - f(x), \tag{3.16}$$

then

$$|e(x)| \leqslant |e_0''|\, h^2 + \tfrac{3}{4}h^3\omega_3(h),$$

hence, using (3.13), we get

$$|e(x)| \leqslant \tfrac{1}{2}h^2\omega_3(h) + \tfrac{3}{4}h^3\omega_3(h),$$

thus

$$|e(x)| \leqslant \tfrac{1}{4}h^2(2+3h)\omega_3(h). \tag{3.17}$$

Now to prove (3.3), it can be easily verified that the result for $e(x)$ on $[x_1, x_{N+1}]$ is based on the result for e_1, and for $e'(x)$ on $[x_1, x_{N+1}]$. But, for $x \in [x_i, x_{i+1}]$

$$e(x) = \int_{x_i}^{x} e'(r)\,\mathrm{d}r + e_i; \quad (i=1(1)N),$$

and since $|e_i| \leqslant |e_1|$, $(i = 1(1)N+1)$, then

$$|e(x)| \leqslant \tfrac{5}{2}h^3\omega_3(h) + |e_1|,$$

consequently, using (3.17), we obtain

$$|e(x)| \leqslant \tfrac{5}{2}h^3\omega_3(h) + \tfrac{1}{2}h^2\omega_3(h) + \tfrac{3}{4}h^3\omega_3(h),$$

thus,

$$|e(x)| \leqslant \tfrac{1}{4}h^2(2 + 13h)\omega_3(h),$$

and (3.3) now follows.

For $x \in [x_0, x_1]$ we have (using (2.1))

$$\begin{aligned} |e''(x)| &\leqslant \frac{12}{h^2}|e_1| + |e_0''| + 4h\omega_3(h) \\ &\leqslant \frac{12}{h^2}\left[\frac{h^2}{2}\omega_3(h) + \frac{3}{4}h^3\omega_3(h)\right] + \frac{1}{2}\omega_3(h) + 4h\omega_3(h), \end{aligned} \tag{3.18}$$

which proves (3.4), if $f \in C^3[0,1]$. Thus, for $x \in [x_0, x_1]$,

$$e'(x) = \int_0^x e''(r)\,\mathrm{d}r,$$

and (3.5) follows immediately.

Next, if $f^{(3)} \in BV[0,1]$, and $x \in [x_0, x_1]$, then

$$|e(x)| \leqslant |e_0''|\,h^2 + \tfrac{3}{4}h^3\omega_3(h) \tag{3.19}$$

or

$$|e(x)| \leqslant \tfrac{1}{2}h^3\,Var(f^{(3)}) + \tfrac{3}{4}h^3\omega_3(h).$$

For $x \in [x_i, x_{i+1}]$ $(i = 1(1)N)$, we have

$$|e(x)| \leqslant \tfrac{5}{2}h^3\omega_3(h) + \tfrac{1}{2}h^3\,Var(f^{(3)}) + \tfrac{3}{4}h^3\omega_3(h)$$

and

$$|e(x)| \leqslant \tfrac{13}{4}h^3\omega_3(h) + \tfrac{1}{2}h^3\,Var(f^{(3)}).$$

Similarly, for $x \in [x_0, x_1]$, we have

$$\begin{aligned} |e''(x)| &\leqslant \frac{12}{h^2}|e_1| + |e_0''| + 4h\omega_3(h) \\ &\leqslant \frac{12}{h^2}\left[\frac{h^3}{2}Var(f^{(3)}) + \frac{3}{4}h^3\omega_3(h)\right] + \frac{h}{2}Var(f^{(3)}) + 4h\omega_3(h), \\ &\leqslant 13h\omega_3(h) + \tfrac{13}{2}h\,Var(f^{(3)}). \end{aligned} \tag{3.20}$$

Since, for $x \in [x_0, x_1]$

$$e'(x) = \int_0^x e''(r)\,\mathrm{d}r,$$

and

$$|e'(x)| \leqslant 13h^2\omega_3(h) + \tfrac{13}{2}h^2\,Var(f^{(3)}),$$

and the proof of Theorem 1 is now completed.

4. Illustrative examples and conclusions

In this section we introduce two test examples, where the notation $1.52 \times 10^{-3} = 1.52(-3)$ is used for convenience.

The algorithm used will be as follows:

- Set $s_0 = f_0$ and $s_{N+1} = f_{N+1}$.
- Use (2.4) to compute s_0''.
- Use (2.3) to compute s_i, $i = 1(1)N$.
- Use (2.1) and (2.2) to compute $s(x)$ at $N+1$ equally spaced points in each subinterval $[x_i, x_{i+1}]$, $i = 1(1)N$.
- $s'(x)$ and $s''(x)$ are computed from (2.1).

Example 1. $f(x) = \frac{1}{2}x^2 - \sin(\pi x)$, in [0, 1].
Table 1 gives the results for $h = \frac{1}{N+1}$, $N = 9, 19, 49, 99, 199$ where $s''(1) = f''(1) = 1$.

Example 2.

$$f(x) = \begin{cases} \cos(2\pi x); & x \in [0, \frac{1}{2}], \\ 2\pi^2(x - \frac{1}{2})^2 - 1; & x \in [\frac{1}{2}, 1]. \end{cases}$$

Table 2 gives the results for $h = 1/(N+1)$, $N = 9, 19, 49, 99, 199$. Recall here that $f^{(3)} \in BV[0, 1]$, and $s''(1) = f''(1) = 4\pi^2$. The results show an agreement with that of Theorem 1.

In summary, we have formulated a specific quartic spline $s \in C^2[0, 1]$ that fits the first and second derivatives (except at 0) of a smooth function at uniform mesh points and the function values at the end points of the interval, together with the condition that $s''(1) = f''(1)$.

Table 1
Maximum absolute errors for Example 1

h	$\|s - f\|$	$\|s' - f'\|$	$\|s'' - f''\|$	$\|e_0''\|$
0.100	1.35(−5)	7.85(−5)	2.41(−3)	2.17(−13)
0.050	8.46(−7)	4.96(−6)	3.05(−4)	8.72(−13)
0.020	2.16(−8)	1.27(−7)	1.95(−5)	7.59(−12)
0.010	1.35(−9)	7.96(−9)	2.44(−6)	4.74(−11)
0.005	8.45(−11)	4.98(−10)	3.06(−7)	8.74(−11)

Table 2
Maximum absolute errors for Example 2

h	$\|s-f\|$	$\|s'-f'\|$	$\|s''-f''\|$	$\|e_0''\|$
0.100	2.62(−3)	2.54(−1)	1.52(+1)	0.52(−1)
0.050	1.66(−4)	3.23(−2)	3.87(0)	1.30(−1)
0.020	4.27(−6)	2.07(−3)	6.22(−1)	2.07(−2)
0.010	2.67(−7)	2.59(−4)	1.55(−1)	5.19(−3)
0.005	1.67(−8)	3.24(−5)	3.89(−2)	1.29(−3)

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